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PSYCHOLOGY AND SCIENTIFIC METHODS

CONCERNING MULTIPLE INTERPRETATIONS OF POSTULATE SYSTEMS AND THE "EXISTENCE" OF HYPERSPACE

WHAT do we mean when we speak of n -dimensional space and n -dimensional geometry, where n is greater than 3? The question refers to talk about space and geometry that are n -dimensional in *points*, for ordinary space, as is well known, is 4-dimensional in lines, 4-dimensional in spheres, 5-dimensional in flat line-pencils, 6-dimensional in circles, etc., and there is naturally no mystery involved in speaking of these latter varieties of multi-dimensional manifolds and their geometries, no matter how high the dimensionality may be. No mystery for the reason that in these geometries everything lies within the domain of intuition in the same sense in which everything in ordinary (point) geometry lies in that domain. In other words, these n -dimensional geometries are nothing but theories or geometries of *ordinary* space, that arise when we take for element, not the point, but some other entity, as the line or the sphere, . . ., whose determination in ordinary space requires more than 3 independent data. Of these varieties of n -dimensional geometry, the inventor was Julius Plücker (d. 1868), but Plücker declined to concern himself with spaces and geometries of more than four dimensions in *points*.

Since Plücker's time, however, such hyper-theories of points have invaded not only almost every branch of pure mathematics, but also—strangely enough—certain branches of physical science, as, for example, the kinetic theory of gases. As to the manner of this latter invasion a hint may be instructive. Given N gas molecules enclosed, say, in a sphere. These molecules are, it is supposed, flying about hither and thither, all of them in motion. Each of them depends on six coordinates, x, y, z, u, v, w , where x, y, z are the usual positional coordinates of the molecule regarded as a point in ordinary space, and u, v, w are the components of the molecule's velocity along the three coordinate axes. Knowing the six things about a given mole-

cule, we know where it is and the direction and rate of its going. The N molecules making up the gas depend on $6N$ coordinates. At any instant these have definite values. These values together define the "state" of the gas at that instant. Now these $6N$ values are said to determine a point in space of $6N$ dimensions. Thus is set up a one-one correspondence between such points and the varying gas states. As the state of the gas changes, the corresponding point generates a locus in the space of $6N$ dimensions. In this way the behavior or history of the gas gets geometrically represented by loci in the hyperspace in question.

Is such geometric n -dimensional phraseology merely a geometric way of speaking about non-spatial things? Even if there exists a space, S_n , one *may* employ the language appropriate to the geometry of the space without having the slightest reference to it, and, indeed, without knowing or even enquiring whether it exists. This use of geometric speech in discourse about non-spatial things is not only possible, but in fact very common. An easily accessible example of it may be found in Bôcher¹ where, in speaking of a set of values of n independent variables as a point in space of n dimensions, the reader is told that the author's use of geometric language for the expression of algebraic facts is due to certain advantages of that language compared with the language of algebra or of analysis; he is told that the geometric terms will be employed "in a wholly conventional algebraic sense" and that "we do not propose even to raise the question whether in any geometric sense there is such a thing as space of more than three dimensions."

It is held by many, including perhaps the majority of mathematicians, that there are no hyperspaces of points and that n -dimensional geometries are, rightly speaking, not geometries at all, but that the facts dealt with in such so-called geometries are nothing but algebraic or analytic or numeric facts expressed in geometric language. If this opinion be correct, then the extensive and growing application of geometric language to analytical theories of higher dimensionality indicates a high superiority of geometric over analytic speech, and it becomes a problem for psychology to ascertain whether the mentioned superiority is adequate to explain the phenomenon in question and, if it be adequate, to show wherein the superiority resides.

No doubt geometric language has a kind of esthetic value that is lacking in the speech of analysis, for the former, being transfused with the rich reminiscences of sensibility, constantly awakens a delightful sense, as thinking proceeds, of the colors, forms, and motions of the sensuous world. This is an emotional value. No doubt, too,

¹"Introduction to Higher Algebra," page 9.

geometric language has, in its distinctive conciseness, an economic superiority, as when, for example, one speaks of the points of the 4-dimensional sphere, $x^2 + y^2 + z^2 + w^2 = r^2$, instead of speaking of the various systems of values of the variables x, y, z, w that satisfy the equation $x^2 + \dots = r^2$. Additional advantages of geometric over analytic speech are brought to light in the following remarks by Poincaré in his address, "L'Avenir des Mathématiques" (1908) :

"Un grand avantage de la géométrie, c'est précisément que les sens y peuvent venir au secours de l'intelligence, et aident à deviner la route à suivre, et bien des esprits préfèrent ramener les problèmes d'analyse à la forme géométrique. Malheureusement nos sens ne peuvent nous mener bien loin, et ils nous faussent compagnie dès que nous voulons nous envoler en dehors des trois dimensions classiques. Est-ce à dire que, sortis de ce domaine restreint où ils semblent vouloir nous enfermer, nous ne devons plus compter que sur l'analyse pure et que toute géométrie à plus de trois dimensions est vaine et sans objet? Dans la génération qui nous a précédés, les plus grands maîtres auraient répondu 'oui'; nous sommes aujourd'hui tellement familiarisés avec cette notion que nous pouvons en parler, même dans un cours d'université, sans provoquer trop d'étonnement.

"Mais à quoi peut-elle servir? Il est aisé de le voir: elle nous donne d'abord un langage très commode, qui exprime en termes très concis ce que le langage analytique ordinaire dirait en phrases prolixes. De plus, ce langage nous fait nommer du même nom ce qui se ressemble et affirme des analogies qu'il ne nous permet plus d'oublier. Il nous permet donc encore de nous diriger dans cet espace qui est trop grand pour nous et que nous ne pouvons voir, en nous rappelant sans cesse l'espace visible qui n'en est qu'une image imparfaite sans doute, mais que en est encore une image. Ici encore, comme dans tous les exemples précédents, c'est l'analogie avec ce qui est simple qui nous permet de comprendre ce qui est complexe."

The question of determining the comparative advantages and disadvantages of the languages of geometry and analysis is a very difficult one. It is evidently in the main a psychological problem. It appears that no serious and systematic attempt has ever been made to solve it. Here, it seems, is an inviting opportunity for a properly qualified psychologist, it being understood that proper qualification would include a familiar knowledge of the languages in question. The interest and manifold utility of such a study are obvious. In the course of such an investigation it would probably be found that the superiority of geometric over analytic speech is alone sufficient to account for the extensive and rapidly increasing literature of what is called n -dimensional geometry and that, in order to account for the

rise of such literature, it is therefore not necessary to suppose the existence of n -dimensional spaces, S_n , the facts dealt with in the literature being, it could be supposed, nothing but analytic facts expressed in geometric language.

If such a result were found, would it follow that S_n does not exist and that consequently n -dimensional geometry must be nothing but analysis in geometric garb? The answer is, no; for we may and we often do assign an adequate cause of a phenomenon or event without assigning the actual cause; and so the possibility would remain that n -dimensional geometry has an appropriate object or subject, namely, a space S_n which, though without sensuous existence, yet has every kind of existence that may warrantably be attributed to ordinary geometric space, S_3 . For this last, though it is imitated by (or imitates) sensible space, as an ideal model or pattern is imitated by (or imitates) an imperfect copy, it is not identical with it. S_3 is not tactile space, nor visual space, nor that of muscular sensation, nor the space of any other sense, nor of all the senses—it is a conceptual space; and whether there are or are not spaces S_4 , S_5 , etc., which have every sort of existence rightly attributable to ordinary geometric space, S_3 , and which differ from the latter only in the accident of dimensionality and in the further accident that S_3 appears in the rôle of an ideal prototype for an actual sensible space, whilst S_4 , S_5 , etc., do not present such an appearance,—that is the question which remains for consideration.

A friend called at my study, and, finding me at work, asked, “What are you doing?” My reply was: “I am trying to tell how a world which probably does not exist would look if it did.” I had been at work on a chapter of what is called 4-dimensional geometry. The incident occurred ten years ago. The reply to my friend no longer represents my conviction. Subsequent reflection has convinced me that a space, S_n , of four or more dimensions has every kind of existence that may be rightly ascribed to the space, S_3 , of ordinary geometry.

The following paragraphs present—merely in outline, for space is lacking for a minute presentation—the considerations that have led me to the conclusion above stated.

Let sensible space be denoted by sS_3 . We know that sS_3 is discontinuous (in the mathematical sense of the term) and that it is irrational. By saying that it is irrational I mean what common experience as well as the results of experimental psychology prove: that three sensible extensions of a same type, let us for definiteness say three sensible lengths, l_1 , l_2 , l_3 , may be such that

$$(1) \quad l_1 = l_2, \quad l_2 = l_3, \quad l_1 \neq l_3.$$

Because sS_3 is thus irrational, because it is radically infected with such contradictions as (1), this space is not, and can not be, the subject or object of geometry, for geometry is rational: it does not admit three such extensions as those in (1). Not only do such contradictions as (1) render sS_3 impossible as a subject or object of geometry, but, when encountered, they produce intolerable intellectual pain—nay, if they could not in some wise be transcended or overcome, they would produce intellectual death, for, unless the law of non-contradiction be preserved, concatenative thinking, the life of intellect, must cease. In case of intellect we may say that its struggle for existence is a struggle against contradictions. But mere existence is not the characteristic aim or aspiration of intellect. Its aim, its aspiration, its joy, is compatibility. Indeed, intellect seems to be controlled by two forces, a *vis a tergo* and a *vis a fronte*: it is driven by discord and drawn by concord. Intellect is a perpetual suitor, the object of the suit being harmony, the beautiful daughter of the muses. Its perpetual enemy is the immortal demon of discord, ever being overcome, but never vanquished.

The victory of intellect over the characteristic contradictions inherent in sS_3 is won through what we call conception. That is to say that either we find or else we create another kind of space which, in order to distinguish it nominally and symbolically from sS_3 , we may call conceptual space, and denote by cS_3 . Unlike sS_3 , cS_3 is mathematically continuous and it is rational. Like sS_3 , cS_3 is extended, it has room, but the room and the extensions are not sensible, they are conceptual; and these extensions are such that, if l_1, l_2, l_3 be three amounts of a given type of extension, as length, say, and if $l_1 = l_2$ and $l_2 = l_3$, then $l_1 = l_3$. The space cS_3 , whether we regard it as found by the intellect or as created by it, is the subject or object of geometry. The current vulgar confusion of sS_3 and cS_3 is doubtless due to the fact that the former imitates the latter, or the latter the former, as a sensible thing imitates its ideal, or as an ideal (of a sensible thing) may be said to imitate that thing; for it is precisely such alternative or mutual imitation that enables us in a measure to control the sensible world through its conceptual counterpart; and so the exigencies of practical affairs and the fact that reciprocally imitating things each reminds us of the other cooperate to cause the sensible and the ideal, the perceptual and the conceptual, to mingle constantly and to become confused in that part of our mental life that belongs to the sensible and the conceptual worlds of three dimensions. Nevertheless, it is a fact to be borne in mind that cS_3 is a subject or object of geometry and that sS_3 is not.

Now, in order to construct the geometry in question, we start with a suitable system of postulates or axioms expressing certain rela-

tions among what are called the elements of cS_3 . These postulates, together with such propositions as are deducible from them, constitute the geometry of cS_3 . I shall call it *pure* geometry, for a reason to be given later, and shall denote it by pG_3 . For definiteness let us refer to the famous and familiar postulates of Hilbert. Any other system would do as well. In the Hilbert system, the elements are called points, lines, and planes. It is customary and just to point out that the terms point, line, and plane are not defined, and in critical commentary it is customary to add:

(A) That, consequently, these terms may be taken to be the names of any things whatsoever with the single restriction that the things must satisfy the relations stated by the postulates;

(B) That, when some admissible or possible interpretation I has been given to the element-names, the postulates P together with their deducible consequences C constitute a definite theory or doctrine D ;

(C) That replacing I by a different interpretation I' produces no change whatever in D ;

(D) That this invariant D is Euclidean geometry of three dimensions; and

(E) That, if we are to speak of D as a theory or geometry of a space, this space is nothing but the ensemble of any kind of things that may serve for an interpretation of P .

That the view expressed in that so-called "critical commentary" does not agree with common sense or with traditional usage is obvious. That it will not bear critical reflection can, I believe, be made evident. Let us examine it a little. In order to avoid the prejudicial associations of the terms point, line, and plane, we may replace them by the terms "roint," "rine," and "rane," so that the first postulate, or axiom, as Hilbert calls it, will read: *Two distinct roints always completely determine a rine*. Or, better still, we may replace them by the symbols e_1 , e_2 , e_3 , so that the reading will be: *Two distinct e_1 's always completely determine an e_2* ; and similarly for the remaining postulates.

We will suppose the phrasing of (A), (B), (C), (D), (E), slightly changed to agree with the indicated new phrasing of the postulates.

It seems very probable that there are no termless relations, *i. e.*, relations that do not relate. It seems very probable that a relation to be a relation must be something actually connecting or subsisting between at least two things or terms. A postulate expressing a relation having terms is at all events ostensibly a statement about the terms, and so it would seem that, if the relation be supposed to be termless, the statement ceases to be a statement about something and, in so ceasing, ceases to be a statement that is true or else is false. In

discourse, it is true, there is frequent seeming evidence that relations are often thought of as termless, as when, for example, we speak of "a relation *and* its terms"; but then we speak also of a neckless fiddle without intending to imply by such locution that there can be a fiddle without a neck. As, however, we do not wish the validity of the following criticism to depend on the denial of the possibility of termless relations, the discussion will be conducted in turn under each of the alternative hypotheses: (h_1) There are termless relations; (h_2) There are no termless relations. We will begin with

HYPOTHESIS h_2

To (A) we make no objection.

Let us now suppose given to P some definite interpretation I . Let us grant that we now have a definite doctrine D , consisting of P and C . Either the things which in I the e 's denote have or they have not content, character, or meaning, m , in excess of the fact that they satisfy P .

(1) Suppose they have not an excessive meaning m . Denote the interpretation by I_1 and the doctrine by D_1 . This D_1 is a queer doctrine. We may ask: what does D_1 relate or refer to? That is, what is it a doctrine of or about? The question seems to admit of no intelligent or intelligible answer. For if the doctrine is about something, it is, it seems natural to say, a doctrine about the I_1 -things (denoted by the e 's); but, by (1), these I_1 -things can not be characterized or indicated otherwise than by the fact of their satisfying P ; and so it appears that such attempted natural answer is reducible and equivalent to saying (a) that the doctrine D_1 is about the things which it is about. In order not to be thus defeated, one might try to give an informing answer by saying that D_1 is a doctrine, not about the I_1 -things, *i. e.*, not about terms of relations, but about the relations themselves. Such an answer is suspicious on account of its unnaturalness, and it is unnatural because the propositions of D_1 wear the appearance of talking explicitly, not about relations, but about terms of relations. Moreover, the answer is not an informing one unless the relations that the doctrine D_1 is alleged to be about can be characterized otherwise than by the fact of their being satisfied by the I_1 -things, for, if they can not be otherwise characterized, evidently by (1) the answer reduces to a form essentially like that of (a). May not one escape by saying that the relations which D_1 is alleged to be a doctrine about are just the relations expressed by the propositions in D_1 ? Does this attempted characterization make the answer in question an informing one? If D_1 is a doctrine about the relations expressed by its propositions, then D_1 says or teaches some-

thing about these relations, for every doctrine, if it be about something, must teach or say something about that which it is about. In the case supposed, what does D_1 teach about the relations? Nothing except that they are satisfied by the I_1 -things. In other words, what D_1 teaches about the relations expressed by its propositions is, by (1), that these are satisfied by things that satisfy them—a not very nutritious lesson. It is possible to make a yet further attempt so to indicate the relations as to render the answer, that the doctrine D_1 is about relations, an informing one. It is known that P may receive an interpretation I' different from I_1 in that the I' -things do not satisfy (1), but have an excessive content, character, or meaning m . May we not give the required indication of the relations that D_1 is said to be a doctrine about by saying that they are relations satisfied by the I' -things, the presence of the m involved making the indication genuine or effective? It seems so at first. But if again we ask what D_1 teaches about the relations thus indicated, we are led into the same difficulty as above. Moreover, when we ask what D_1 is a doctrine about we expect an answer in terms in somewhat mentioned or intimated in the D_1 discourse, whilst in the case in hand the required indication has depended on m , a thing expressly excluded from the D_1 discourse by (1).

So, I repeat, D_1 is a queer doctrine.

It must be aded that if there be an interpretation I_1 , it is unique of its kind; for if I_1' were an interpretation satisfying (1), the I_1' -things would have no excessive meaning m ; hence they would be simply the I_1 -things, and I_1 and I_1' would be merely two symbols for a same interpretation.

Accordingly, if there were an interpretation I_1 , but no other, *i. e.*, no interpretation I in which the I -things did not satisfy (1), then (C) would be pointless; by (D), D_1 would be Euclidean geometry of three dimensions; and, by (E), Euclidean space, if we wished to speak of D_1 as a geometry of a space, would be the ensemble of the I_1 -things; but, if we wished to characterize the I_1 -things, the elements of Euclidean space, we could only say that they are the things satisfying certain relations, and, if we wished to indicate what relations, we could only say, the relations satisfied by those things: a very handsome circle.

In the following it will be seen that we are in fact not imprisoned within that circle.

(2) Suppose the I -things of the above-assumed interpretation I do not satisfy (1), but have an excessive meaning m . (It is known that such an I is possible, an example being found by taking for an e_1 any ordered triad of real numbers (x, y, z) ; for an e_2 the ensemble

of triads satisfying any two distinct equations,

$$A_1x + B_1y + C_1z + D_1 = 0, \quad A_2x + B_2y + C_2z + D_2 = 0,$$

in neither of which the coefficients are all of them zero; and for an e_3 the ensemble of triads satisfying any one such equation; the presence of m being evident in countless facts such as the fact, for example, that an e_1 is composed of numbers studied by school-boys or useful in trade without regard to their ordered triadic relationship.) Denote the assumed definite interpretation I by I_2 to remind us that it satisfies (2), and denote the corresponding doctrine by D_2 . It is immediately evident that there is an interpretation I_1 and hence a doctrine D_1 , for to obtain I_1 it is sufficient to abstract from the m of the I_2 -things and to take the abstracts (which plainly satisfy (1)) for I_1 -things.

Are D_1 and D_2 but two different symbols for one and the same doctrine, as asserted by (C)? Evidently not. For, in respect of D_2 , we can give an informing answer to the question, what is D_2 a doctrine about? Owing to the presence of the m in the I_2 -things, the answer will be an informing one whether it be the natural answer that D_2 is a doctrine about the I_2 -things, or one of the less natural answers, that D_2 is about the relations having the I_2 -things for terms, that D_2 is about the relations expressed by its propositions; whilst, as we have seen, owing to the absence of m in the I_1 -things no such answers were, in respect of D_1 , informing answers.

Can not (C) be saved by refusing to admit that there is an interpretation I_1 , and so refusing to admit that there is a D_1 ? If there is no I_1 and hence no D_1 , then (C) is pointless unless there is an I_2' and so a D_2' in which the I_2' -things have an m' different from the m of the I_2 -things. But if there is an I_2' thus different from I_2 , then obviously D_2' and D_2 are, contrary to (C), different doctrines, for they are respectively doctrines about the I_2 -things and the I_2' -things, and these thing-systems are different by virtue of the difference of m and m' . Now, it is known that there are two such differing interpretations I_2 and I_2' . For we may suppose I_2 to be the possible interpretation indicated in the above parenthesis. And for I_2' we may take for e_1 any ordered triad of real numbers, except a specified triad (a, b, c), and including (∞, ∞, ∞); for e_2 the ensemble of triads, except (a, b, c), that satisfy any pair of equations,

$$\begin{aligned} A_1(x^2 + y^2 + z^2) + 2B_1(x - a) + 2C_1(y - b) \\ + 2D_1(z - c) - A_1(a^2 + b^2 + c^2) = 0, \\ A_2(x^2 + y^2 + z^2) + 2B_2(x - a) + 2C_2(y - b) \\ + 2D_2(z - c) - A_2(a^2 + b^2 + c^2) = 0; \end{aligned}$$

and for e_3 the ensemble satisfying any one such equation. Just as

when we compared D_1 and D_2 , so here the conclusion is, that (C) is not valid.

As a matter of fact mathematicians know that there are possible infinitely many different interpretations of P . It follows from the foregoing that there are correspondingly many different doctrines. For the sake of completeness we may include D_1 among these, although, for the purpose of answering a hypothetical objection, we momentarily supposed D_1 to be disputable or inadmissible.

Which one of the D 's is (or should be called) Euclidean geometry of three dimensions? I say which "one"? For, as no two are identical, it would be willful courting of ambiguity to allow that two or more of them should be so denominated. Which *one*, then? Evidently not one of the numerical ones, such, for example, as the two above specified. For who has ever *really* believed that a point, for example, *is* a triad of numbers? We know that the Greeks did arrive at geometry; we know that they did not arrive at it through numbers; and we know that, in their thought, points were not number triads, nor were planes and lines, for them, certain ensembles of such triads. The confusion, if anybody ever was really thus confused, is due to the *modern* discovery that number triads and certain ensembles of them happen to satisfy the same relations as the Greeks found to be satisfied by what they called points, lines, and planes. There is really no excuse for the confusion, for, if Smith is taller than Brown, and yonder oak is taller than yonder beech, it obviously does not follow that Smith is the oak and Brown the beech.

Evidently Euclidean geometry of three dimensions is that particular D for which the I -things are points, lines, and planes. Here it is certain to be asked: What, then, are points, lines, and planes? And the asker will mean to imply that, in order to maintain the proposition, it is necessary to define these terms. The proper reply is that it is not necessary to *define* them. All that can be reasonably required is that they be indicated, pointed out, sufficiently described for purposes of recognition, for what we desire is to be able to say or to recognize what Euclidean geometry is about. To the question one might, not foolishly, reply that the terms in question denote things that you and I, if we have been disciplined in geometry, converse understandingly about when we converse about geometry, though neither of us is able to say with *absolute precision* what the terms mean. For who does not know that it is possible to write an intelligent and intelligible discourse about cats, for example, without being able to tell (for who can tell?) precisely what a cat is? And if it be asked what the discourse is about, who does not know that it is an informing answer to say that it is about cats. It is informing because the term cat has an excessive meaning, a meaning

beyond that of satisfying the propositions (or relations) of the discourse.

Just here it is well worth while to point out an important lesson in the procedure of Euclid. Against Euclid it is often held as a reproach that he attempted to define the element-names, point, line, and plane, since no definitions of them could render any logical service, that is, in the strictly deductive part of the discourse. But to render no logical service is not to render no service. And the lesson is that the definitions in question, which it were perhaps better to call descriptions, do render an extralogical service. They render such service not only in guiding the imagination in the matter of invention, but also in serving to indicate, with a goodly degree of success, the excessive meaning *m* of the elements denoted by the terms in question and in thus serving to make known what it is that the deductive part of the discourse is about. One should not forget that no discourse, no doctrine, not even so-called pure logic itself, is exclusively deductive, for in any doctrine there is reference, implicit or explicit, to something extraductive or extralogical, reference, that is, to something which the doctrine is about.

Are the three Euclidean "definitions," thus viewed as descriptions, sufficient or adequate to the service that they are here viewed as rendering? If by sufficient or adequate be meant exhaustive, the answer is, of course, no. For we may confidently say that no possible description, that is, no description involving only a finite number of words, possibly can exhaust the meaning of a system of terms except, possibly, in the special case where these have no meaning beyond what they must have in order merely to satisfy a finite number of postulates. But exhaustive is not what is meant by adequate. To employ a previous illustration, it is not necessary to give or to attempt an exhaustive description of "cat" in order to tell adequately what it is that a discourse ostensibly about cats is ostensibly about. It is a question of intent. A description is nearly, if not quite, adequate if it enables us to avoid thinking that terms are intended to denote what they are not intended to denote. And, whilst we may not admit that the three Euclidean "descriptions" are the best that can be invented for the purpose, yet we must allow that they have long served the end in question pretty effectively and that they are qualified to continue such service. They have been and they are good enough, for example, to save us from thinking that the things which in geometry have been denoted by the terms, points, lines, and planes, are identical with number triads, etc. The open secret of their thus saving us is no doubt in their causing us to think of points, lines, and planes in terms of, or in essential connection with, what we know as *extension*, whilst numbers and number

ensembles are not things naturally so conceived. For evidently the notions of "length" and "breadth" involved in the Euclidean "descriptions" are not *metric* in meaning; they do not signify definite or numeric quantities or amounts of something (as when we say the length of this or that thing is so and so much); but plainly they are generic notions connoting *extension*. It is safe to say that a mind devoid of the concept or the sense of extension could not know what things the "descriptions" aim at describing. It is true that Euclid's "description" of a point as "that which has no part" implies a *denial* of extension, but the denial is one of *extension*, and, in its contextual atmosphere, it is *felt* to be essential to an adequate indication of what is meant by point. On the other hand, if one were (and how unnatural it would be!) to describe an ordered triad of numbers as "that which has no part," it would be immediately necessary to explain away the seeming falsity of the description by saying that the triad is not the ordered multiplicity (of three numbers) *as* a multiplicity, but is merely the *uniphase* of the multiplicity, and that it is this uniphase which has no part. If, next, we were to say that thus extension is denied to the uniphase, the statement, though true, would be *felt* to be inessential to an adequate indication of what is meant by a triad of numbers. Such felt difference is alone sufficient to make any one pause who is disposed to adopt the current creed that a point is nothing but an ordered triad of numbers. It is not contended that a point is *composed* of extension; the contention is that point and extension are so connected that a mind devoid of the latter notion would be devoid of the former, just as a mind devoid of the notion of variable or variation would be devoid of the notion of constant, though a constant is not a thing *consisting* of variation; just as the notion of limit would not be intelligible except for the notion of something that may have a limit, though the limit is not *composed* of it; and just as an instant, which is not *composed* of time, would not be intelligible except for the notion of time.

In a discussion of such matters it is foolish and futile to talk about "proofs." The question, as said, is one of intent; it is a question of self-veracity, of getting aware of and owning what it is that we mean by the terms and symbols of our discourse. If, despite the Euclidean "descriptions" and despite any and all others that may supplement or supplace them, one fails to see that *extension* is essentially involved in the meaning that the terms points, lines, and planes, are intended to have, the failure will be because "as the eyes of bats are to the blaze of day, so is the reason in our soul to the things which are by nature most evident of all." Nothing is more evident than that there is something that is called extension. We have but to open our eyes to get aware that we are behold-

ing an *expanse*, something extended. We see things *as* extended; things *as* extended are revealed to the tactile sense; a region or room involving extension is a datum of the muscular sensations connected with our bodily movements; and so on. So much is certain. But it is said and rightly said that these are *sensible* things; that the extension they are revealed as having is *sensible* extension; that these sensibles are infected with contradiction, above noted, revealed in common experience, and confirmed by the psychophysical law of Weber and Fechner; that geometry is free from contradiction; that, therefore, geometry is not a doctrine about these sensibles; that among these sensibles are not the things which in geometry are denoted by the terms point, line, and plane; and that, if these terms imply or connote extension, as asserted, this extension is not sensible extension. Granted. The "connoted extension" is not sensible, it is conceptual. How know, however, that there is conceptual extension? The answer is, by arriving at it. (We need not here debate whether such "arriving" is best called creating or is best called finding.) But how does the mind arrive at it? By doing certain things to the sensibles, the raw material of mental architecture. What things? An exhaustive answer is unnecessary—perhaps impossible. The things are of two sorts: the mind gives to the sensibles; it takes away from them. Consider, for example, a sensible line. From it the conceptualizing intellect takes away (abstracts from, disregards) certain things that the sensible in question has or may have, as color, weight, temperature, etc., including *part* of the extension, thus, I mean, narrowing and thinning away all breadth and thickness. What of the extension called length? Have the narrowing and thinning taken it away? It was not so intended, the opposite was intended. Yet no sensible length (extension) remains. Does the narrowing and thinning involve shortening? We are absolutely certain that it does not. What, then, is it that has happened? Evidently that, by the indicated taking away, the mind has arrived at insensible length, one kind of insensible extension, that is, at conceptual length, one kind of conceptual extension. A stretch, we are sure, remains, but it is not a sensible stretch. The extension thus arrived at is yet not the extension connoted by or involved in the things that geometry is about, for in the taking-away process of arriving at it there is nothing to disinfect it of the contradictions inherent in the sensible with which we started. It remains, then, to follow the indicated process of taking away by a process of giving, that is to say, it remains to endow the conceptual extension (arrived at) with continuity so as to render it free from the mentioned contradictions. This done, the kind of extension meant in ordinary geom-

etry or ordinary geometric space is arrived at. Such is, in kind, the conceptual extension that, it is here held, is essential to what the geometric terms, point, line, plane, are intended to mean. Without further talk we may say that such extension is essential in the conceptual space that, we may say, ordinary Euclidean geometry is about in being about the elements of the space.

If we denote this conceptual space by cS_3 to distinguish it from (non-geometrizable) sensible space sS_3 , then the geometry of cS_3 , if constructed by means of postulates P making no indispensable use of algebraic analysis, may be called pure geometry, pG_3 . If, as in the Cartesian method, we use ordered number triads, etc., as we may use them, not to be points, etc., but to represent points, etc., then we get analytical geometry, aG_3 , of cS_3 . On the other hand, if, as we may, we interpret the P by allowing the I -things to be number triads, etc., as above indicated, the resulting doctrine is, not geometry, but a pure algebra or analysis, pA_3 . If we use points, etc., not to be, but to represent, number triads, etc., and so employ geometric language in constructing pA_3 , we get by this kind of anti-Cartesian procedure, not a geometry, but geometrical analysis, gA_3 .

HYPOTHESIS h_1

It is unnecessary to say anything and is not worth while to say much under this hypothesis. For if the e 's in P do not denote something, then as the relations (if there be any) are termless, the doctrine D (if there be one) is not about anything, unless about the relations, but about these it says nothing, for, if it says aught about them, what it says is that they are satisfied by certain terms whose presence in the discourse is excluded by h_1 . We may profitably say, however, that, in the case supposed where the e 's do not denote something but are merely uninterpreted variables ready, so to speak, to denote something—in this case we may say that, though there is no doctrine D , there is a *doctrinal function*, $\Delta(e_1, e_2, e_3)$. Then we should add that the doctrines that do arise from actualized possible interpretations of the e 's are so many *values* of Δ . This function Δ , if we give some warning mark as Δ' to its symbol, may be further conveniently employed in talking about an ambiguous one of the doctrines in question, *i. e.*, about "any value," an ambiguous value, of the function. As above argued, these values, these doctrines are identical in *form*, they are *isomorphic*, all of them having the form of Δ , but no two of them are the same in respect of content, reference, or meaning. In this conclusion, analysis, happily, agrees with traditional usage, intuition, and common sense.

CONCLUDING CONSIDERATIONS

We are, I believe, now prepared to answer definitively the long-vexed question: *What, if any, sort of existence have point spaces of four or more dimensions?*

As we have seen, the conceptual space cS_3 of ordinary geometry is an affair involving extension; it is a triply extended conceptual spread or expanse: three independent linear extensions in it may be chosen; these suffice to determine all the others. So much is as certain as anything can be. It is equally certain that we can, for we do without meeting contradiction, by means of postulates or otherwise, conceive (not perceive or imagine) a quadruply extended spread or expanse, one, that is, in which it is possible to choose four independent linear extensions, and then by reference to these to determine all the rest. There is not the slightest difference in kind among the four independents and not the slightest difference between any three of these and the three of cC_3 . The spread or expanse thus set up is a cS_4 ; like cS_3 , it is purely conceptual; the extension it involves is, in kind, identical with that of cS_3 ; it contains spreads of the type of cS_3 as elements just exactly as a cS_3 contains planes or spreads of type cS_2 as elements; it differs not at all from cS_3 except in being one degree higher in respect of dimensionality. In a word, cS_4 (and, of course, cS_5 , and so on) has the same kind of existence as cS_3 . It is true that cS_3 is "imitated" by our sensible space sS_3 , whilst there is no sS_4 thus imitating cS_4 . But this writing is not intended for one who is capable of thinking that the mentioned sensible imitation or imitability of cS_3 confers upon the latter a new or peculiar kind of existence.

But one thing remains to be said, and it is important. If one denies that cS_3 has the conceptually extensional existence, above alleged, then, of course, the denial extends also to cS_4 , and the two spaces are, in respect of existence, still on a level. If the denier then asserts, and such is the alternative, that cS_3 is only the ensemble of number triads, etc., as above explained, then, if he be right, cS_4 is only, but equally, the ensemble of ordered quatrains, etc., of numbers. Here, again, cS_3 and cS_4 have precisely the same kind of existence. The conclusion is that *hyperspaces have every kind of existence that may be warrantably attributed to the space of ordinary geometry.*

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